

HEREDITARY KONIG EGervary COLLECTIONS

ADI JARDEN

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ABSTRACT. Let G be a simple graph with vertex set $V(G)$. A subset S of $V(G)$ is independent if no two vertices from S are adjacent. The graph G is known to be a Konig-Egervary (KE in short) graph if $\alpha(G) + \mu(G) = |V(G)|$, where $\alpha(G)$ denotes the size of a maximum independent set and $\mu(G)$ is the cardinality of a maximum matching. Let $\Omega(G)$ denote the family of all maximum independent sets.

A collection F of sets is an hke collection if $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha$ holds for every subcollection Γ of F . We characterize an hke collection and invoke new characterizations of a KE graph.

We prove the existence and uniqueness of a graph G such that $\Omega(G)$ is a maximal hke collection. It is a bipartite graph. As a result, we solve a problem of Jarden, Levit and Mandrescu [4], proving that F is an hke collection if and only if it is a subset of $\Omega(G)$ for some graph G and $|\bigcup F| + |\bigcap F| = 2\alpha(F)$.

Finally, we show that the maximal cardinality of an hke collection F with $\alpha(F) = \alpha$ and $|\bigcup F| = n$ is $2^{n-\alpha}$.

1. INTRODUCTION

In this paper we study hereditary KE collections: collections of sets, F , of a fixed cardinality α , such that the equality

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha$$

holds for every non-empty subset Γ of F .

The following theorem is a restatement of [2, Theorem 2.6]:

Theorem 1.1. *G is a KE graph if and only if for some hke subcollection Γ of $\Omega(G)$ there is a matching $M : V(G) - \bigcup \Gamma \rightarrow \bigcap \Gamma$.*

In most of the sections of this paper, the main issue is the hke collections rather than graphs. In Section 2, we present characterizations for an hke collection and invoke a new characterization for a KE graph. Definition 2.7 presents the typical example of a maximal hke collection. Theorem 3.12 characterizes a maximal hke collection by its cardinality. By Corollary 4.10, the typical example for α (that is defined in Definition 2.7) is the unique maximal hke collection, where α is given. By Theorem 5.11, there is a unique KE graph (actually, a bipartite graph), G , such that $\Omega(G)$ is a maximal hke collection. By Theorem 6.3, a collection of sets F is an hke collection if and only if it is included in $\Omega(G)$ for some graph G and $|\bigcup F| + |\bigcap F| = 2\alpha(F)$. By Theorem 7.6, the maximal cardinality of an hke collection, F , with $\alpha(F) = \alpha$ and $|\bigcup F| = n$ is $2^{n-\alpha}$.

2. KONIG EGERVARY COLLECTIONS

In this section, we define a relevant collection, a KE collection and an hke collection. We characterize hke collections and invoke new characterizations for a KE graph.

Definition 2.1. A *relevant collection* is a finite collection of finite sets such that the number of elements in each set in F is a constant positive integer, denoted $\alpha(F)$. When F is clear from the context, we omit it, writing α .

The following definition contradicts the definition of a Konig Egervary collection in [2] (see Definition 6.1).

Definition 2.2. Let F be a relevant collection. F is said to be a *Konig Egervary collection* (KE collection in short), if the following equality holds:

$$|\bigcup F| + |\bigcap F| = 2\alpha.$$

Definition 2.3. An *hereditary Konig Egervary collection* (hke collection in short) is a collection of sets, F , such that for some positive integer, α , the equality

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha$$

holds for every non-empty sub-collection, Γ , of F . We call this α , $\alpha(F)$. We may omit F , where it is clear from the context.

Proposition 2.4. Let F be an hke collection. Clearly, for each $A \in F$, $|A| = \alpha(F)$. So F is a relevant collection.

Proof. By Definition 2.3, where we substitute $\Gamma = \{A\}$. ◊

Proposition 2.5. Let F be a relevant collection. If $|F| \leq 2$ then it is an hke collection.

Proof. It is clear when $|F| = 1$. So Assume $|F| = 2$, $F = \{A, B\}$. Take a non-empty sub-collection Γ of F . Without loss of generality, $\Gamma = F$. So

$$|\bigcup \Gamma| + |\bigcap \Gamma| = |A \cup B| + |A \cap B| = |A| + |B| = 2\alpha(F).$$

◊

Proposition 2.6. Let G be a KE graph. Then $\Omega(G)$ is an hke collection.

Proof. By [4, Theorem 3.6] and [4, Corollary 2.8]. ◊

The following definition presents the typical example of a maximal hke collection for a fixed α : by Proposition 2.8, it is an hke collection, by Theorem 3.12, it is a maximal hke collection and by Theorem 4.9, it is the unique example up to isomorphism.

Definition 2.7. Let α be a positive integer. *The typical collection for α is the collection of subsets S of $[2\alpha]$ such that $i \in S \leftrightarrow i + \alpha \notin S$ holds for every $i \in [\alpha]$.*

Proposition 2.8. *Let α be a positive integer. Let F be the typical collection for α . Then F is an hke collection and $|F| = 2^\alpha$.*

Proof. Clearly, $|F| = 2^\alpha$. We prove that F is an hke collection. Let Γ be a non-empty subset of F . For each $i \in [\alpha]$ we have $i \in \bigcup \Gamma$ if and only if $i + \alpha \notin \bigcap \Gamma$ and $i \in \bigcap \Gamma$ if and only if $i + \alpha \notin \bigcup \Gamma$. Hence

$$|\bigcup \Gamma \cap [\alpha]| + |\bigcap \Gamma \cap ([2\alpha] - [\alpha])| = \alpha,$$

and

$$|\bigcap \Gamma \cap [\alpha]| + |\bigcup \Gamma \cap ([2\alpha] - [\alpha])| = \alpha.$$

Therefore

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha.$$

So F is an hke collection. ◊

In Theorem 2.13 we present equivalent conditions for F being an hke collection. Consider the following equality:

Equality 2.9.

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Proposition 2.10. *Let F be a relevant collection.*

The following are equivalent:

- (1) *F is an hke collection.*
- (2) *Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F ,*
- (3) *Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.*

Before proving Proposition 2.10, we present an exercise:

Exercise 2.11. Assume that $\{A, B, C, D\}$ is an hke collection (so in particular $\{A, B, C\}$ is an hke collection). Prove:

- (1) $|A - B - C| = |B \cap C - A|$. A clue: $A - B - C = (A \cup B \cup C) - (B \cup C)$ and $B \cap C - A = (B \cap C) - (A \cap B \cap C)$.
- (2) $|A \cap B - C - D| = |C \cap D - A - B|$. A clue: $A \cap B - C - D = (A - C - D) - (A - B - C - D)$. Apply Clause (1).

We now prove Proposition 2.10.

Proof. (1) \Rightarrow (2): We prove it by induction on $r = |\Gamma_1|$.

Case a: $r = 1$, so $\Gamma_1 = \{A^*\}$ for some set A^* . In this case, we apply the idea of Exercise 2.11(1).

We should prove that

$$|A^* - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - A^*|,$$

namely,

$$|\bigcup \Gamma_2 \cup A^*| - |\bigcup \Gamma_2| = |\bigcap \Gamma_2| - |\bigcap \Gamma_2 \cap A^*|,$$

or equivalently,

$$|\bigcup \Gamma_2 \cup A^*| + |\bigcap \Gamma_2 \cap A^*| = |\bigcap \Gamma_2| + |\bigcup \Gamma_2|.$$

But by Clause (1), each side of this equality equals 2α .

Case a: $r > 1$. In this case, we apply the idea of Exercise 2.11(2). We fix $A^* \in \Gamma_1$. First we write three trivial equalities, for convenience:

$$\bigcap (\Gamma_1 - \{A^*\}) = \{x : x \in A \text{ for every } A \in \Gamma_1 \text{ with } A \neq A^*\},$$

$$\bigcup (\Gamma_1 - \{A^*\}) = \{x : x \in A \text{ for some } A \in \Gamma_1 \text{ with } A \neq A^*\}$$

and

$$\bigcap (\Gamma_1 \cup \{A^*\}) = A^* \cap \bigcap \Gamma_1.$$

We now begin the computation.

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap (\Gamma_1 - \{A^*\}) - \bigcup \Gamma_2| - |\bigcap (\Gamma_1 - \{A^*\}) - \bigcup (\Gamma_2 \cup \{A^*\})|.$$

The right side of this equality is a subtraction of two summands. We apply the induction hypothesis on each summand:

$$|\bigcap (\Gamma_1 - \{A^*\}) - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup (\Gamma_1 - \{A^*\})|$$

and

$$|\bigcap (\Gamma_1 - \{A^*\}) - \bigcup (\Gamma_2 \cup \{A^*\})| = |\bigcap (\Gamma_2 \cup \{A^*\}) - \bigcup (\Gamma_1 - \{A^*\})|.$$

By the three last equalities we get:

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup (\Gamma_1 - \{A^*\})| - |\bigcap (\Gamma_2 \cup \{A^*\}) - \bigcup (\Gamma_1 - \{A^*\})|.$$

So

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Equality 2.9 is proved, so Clause (2) is proved.

(2) \Rightarrow (1) : Let Γ be a non-empty subset of F . Fix $D \in \Gamma$. Since F is a relevant collection, $|D| = \alpha$ (this is the unique place where we use the assumption that F is a relevant collection, but we eliminate this assumption later). Therefore it is enough to prove that

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2|D|,$$

or equivalently,

$$|\bigcup \Gamma - D| = |D - \bigcap \Gamma|.$$

Let H be the set of ordered pairs $\langle \Gamma_1, \Gamma_2 \rangle$ of non-empty disjoint subsets of Γ such that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $D \in \Gamma_2$.

By Clause (2),

$$\sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

So it is enough to prove the following two equalities:

$$|\bigcup \Gamma - D| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2|$$

and

$$|D - \bigcap \Gamma| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_2 - \bigcup \Gamma_1|.$$

Since their proofs are dual, we prove the first equality only.

$$\bigcup \Gamma - D = \bigcup_{\langle \Gamma_1, \Gamma_2 \rangle \in H} (\bigcap \Gamma_1 - \bigcup \Gamma_2),$$

(on the one hand, if $x \in \bigcup \Gamma - D$ then for $\Gamma_1 = \{A \in \Gamma : x \in A\}$ and $\Gamma_2 = \{A \in \Gamma : x \notin A\}$ we have $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ and $\langle \Gamma_1, \Gamma_2 \rangle \in H$. On the other hand, assume that $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ for some $\langle \Gamma_1, \Gamma_2 \rangle \in H$. Then $x \in \bigcup \Gamma$ (because $x \in \bigcap \Gamma_1$ and $\emptyset \neq \Gamma_1 \subseteq \Gamma$) and $x \notin D$ (because $x \notin \bigcup \Gamma_2$ and $D \subseteq \bigcup \Gamma_2$). So $x \in \bigcup \Gamma - D$). Therefore

$$|\bigcup \Gamma - D| = \sum_{\langle \Gamma_1, \Gamma_2 \rangle \in H} |\bigcap \Gamma_1 - \bigcup \Gamma_2|,$$

because this is a sum of cardinalities of disjoint sets (if $\langle \Gamma_1, \Gamma_2 \rangle$ and $\langle \Gamma_3, \Gamma_4 \rangle$ are two different pairs in H then there is no element $x \in (\bigcap \Gamma_1 - \bigcup \Gamma_2) \cap (\bigcap \Gamma_3 - \bigcup \Gamma_4)$. Otherwise, take $A \in \Gamma_1 - \Gamma_3$ (or vice versa). So $A \in \Gamma_4$. Hence, $x \in \bigcap \Gamma_1 \subseteq A$ and $x \notin \bigcup \Gamma_4 \supseteq A$, a contradiction).

The implication (2) \Rightarrow (1) is proved.

Since Clause (3) is a private case of Clause (2), it remains to prove (3) \Rightarrow (2). Let Γ_1, Γ_2 be two non-empty disjoint subsets of F . We should prove Equality 2.9 for these Γ_1 and Γ_2 , without assuming $\Gamma_1 \cup \Gamma_2 = F$. Let H be the set of disjoint pairs $\langle \Gamma_1^+, \Gamma_2^+ \rangle$ of F such that $\Gamma_1 \subseteq \Gamma_1^+$, $\Gamma_2 \subseteq \Gamma_2^+$ and $\Gamma_1^+ \cup \Gamma_2^+ = F$.

By Clause (3),

$$\sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_2^+ - \bigcup \Gamma_1^+|.$$

So it remains to prove the following two equalities:

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+|$$

and

$$|\bigcap \Gamma_2 - \bigcup \Gamma_1| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_2^+ - \bigcup \Gamma_1^+|,$$

Since their proofs are dual, we prove the first equality only.

$$\bigcap \Gamma_1 - \bigcup \Gamma_2 = \bigcup_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} (\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+)$$

(on the one hand, if $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ then for $\Gamma_1 = \{A \in \Gamma : x \in A\}$ and $\Gamma_2 = \{A \in \Gamma : x \notin A\}$, we have $x \in \bigcap \Gamma_1^+ - \bigcup \Gamma_2^+$ and the pair $\langle \Gamma_1^+, \Gamma_2^+ \rangle$ belongs to H . On the other hand, if $x \in \bigcap \Gamma_1^+ - \bigcup \Gamma_2^+$ for some $\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H$ then $x \in \bigcap \Gamma_1^+ \subseteq \bigcap \Gamma_1$ and $x \notin \bigcup \Gamma_2^+ \supseteq \bigcup \Gamma_2$. Hence, $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$). Therefore

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2| = \sum_{\langle \Gamma_1^+, \Gamma_2^+ \rangle \in H} |\bigcap \Gamma_1^+ - \bigcup \Gamma_2^+|,$$

because it is a sum of disjoint sets. \dashv

The following proposition eliminates the assumption that F is a relevant collection.

Proposition 2.12. *Clause (3) of Proposition 2.10 implies that F is a relevant collection.*

Proof. Define

$$\alpha = \frac{|\bigcup F| + |\bigcap F|}{2}.$$

Let $D \in F$. We prove that $|D| = \alpha$. Let P denote the family of partitions $\{\Gamma_1, \Gamma_2\}$ of F into two non-empty subcollections.

If $x \in \bigcup F$ then $x \in \bigcap \Gamma_1 - \bigcup \Gamma_2$ for some partition $\{\Gamma_1, \Gamma_2\} \in P$ or $x \in \bigcap F$.

Let

$$P_1 = \{\{\Gamma_1, \Gamma_2\} \in P : D \in \Gamma_1\}$$

and

$$P_2 = \{\{\Gamma_1, \Gamma_2\} \in P : D \notin \Gamma_1\}.$$

Define

$$x = \sum_{\{\Gamma_1, \Gamma_2\} \in P_1} |\bigcap \Gamma_1 - \bigcup \Gamma_2|$$

and

$$y = \sum_{\{\Gamma_1, \Gamma_2\} \in P_2} |\bigcap \Gamma_1 - \bigcup \Gamma_2|.$$

By Clause (3) of Proposition 2.10, we have $x = y$.

It is easy to check the following three equalities:

- (1) $|\bigcup F| = x + y + |\bigcap F| = 2x + |\bigcap F|,$
- (2) $|D| = x + |\bigcap F|$ and
- (3) $|\bigcup F| + |\bigcap F| = 2\alpha.$

Hence, $|D| = \alpha$. Since D was an arbitrary set in F , F is a relevant collection. \dashv

Theorem 2.13. *Let F be a collection of sets.*

The following are equivalent:

- (1) *F is an hke collection.*
- (2) *Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F ,*
- (3) *Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.*

Proof. By Proposition 2.10, it is enough to prove that each clause implies that F is a relevant collection. By Proposition 2.4, Clause (1) implies that F is a relevant collection. By Proposition 2.12 Clause (3) implies that F is a relevant collection. But Clause (2) implies Clause (3). \dashv

Corollary 2.14. *Let G be a graph. The following are equivalent:*

- (1) *G is a KE graph.*

- (2) For some non-empty hke collection $F \subseteq \Omega(G)$, there is a matching $M : V[G] - \bigcup F \rightarrow \bigcap F$ and Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F .
- (3) For some non-empty hke collection $F \subseteq \Omega(G)$, there is a matching $M : V[G] - \bigcup F \rightarrow \bigcap F$ and Equality 2.9 holds for every two non-empty disjoint sub-collections, Γ_1, Γ_2 of F with $\Gamma_1 \cup \Gamma_2 = F$.

Proof. By Theorem 2.13 and Theorem 1.1. \dashv

3. MAXIMAL HKE COLLECTIONS

In this section, we study maximal hke collections. A priori it is not clear whether there is a maximal hke collection, but we prove it. Moreover, we prove the uniqueness of a maximal hke collection for a given α and characterize it.

The most simple characterization of a maximal hke collection is its cardinality: $|F| = 2^\alpha$ (see Theorem 3.12). This theorem is proved by the following argument: Let F be a maximal hke collection. We fix a set $A \in F$ and present a bijection $f_A : F \rightarrow P(A)$.

Definition 3.1. Let F be an hke collection. For each $A \in F$, we define a function $f_A : F \rightarrow P(A)$ by $f_A(D) = A \cap D$.

In the next propositions we present several properties of a maximal hke collection:

- (1) by Proposition 3.2, f_A is an injection (actually, this property holds for every hke collection),
- (2) by Proposition 3.4, $\bigcap F = \emptyset$,
- (3) by Proposition 3.10 f_A is a surjection and
- (4) By Theorem 3.12, $|F| = 2^\alpha$.

Proposition 3.2. Let F be an hke collection and let $A \in F$. Then the function f_A is an injection of F into $P(A)$. So $|F| \leq 2^\alpha$.

Proof. Let $A, B, C \in F$ with $A \cap B = A \cap C$. We have to show that $B = C$. By symmetry, it is enough to prove that $C \subseteq B$. But $A \cap B - C = (A \cap B) - (A \cap C) = \emptyset$. So $|C - A - B| = |A \cap B - C| = 0$, namely, $C - B \subseteq A$. Hence, $C - B \subseteq (A \cap C) - (A \cap B) = \emptyset$. \dashv

Proposition 3.3. If F is an hke collection and $|F| = 2^\alpha$ then F is a maximal hke collection.

Proof. By Proposition 3.2 \dashv

Proposition 3.4. Let F be a maximal hke collection. Then $\bigcap F = \emptyset$.

Proof. Fix $A \in F$. Define

$$B = A \cup C - \bigcap F,$$

where C is a set of new elements ($C \cap \bigcup F = \emptyset$) of cardinality $|\bigcap F|$. We now prove that $F' = F \cup \{B\}$ is an hke collection. First note that $|B| = |A| = \alpha$. Let Γ_1 and Γ_2 be two non-empty disjoint subsets of F' . We have to show that Equality 2.9 holds.

If $B \notin \Gamma_1 \cup \Gamma_2$ then it holds, because F is an hke collection. Assume that $B \in \Gamma_1 \cup \Gamma_2$. Since Γ_1 and Γ_2 are disjoint, by symmetry, we may assume that $B \in \Gamma_1 - \Gamma_2$. The proof is separated into four cases, according to the following two questions:

- (1) Is B the only set in Γ_1 ?
- (2) Does A belong to Γ_2 ?

Case a: $\Gamma_1 = \{B\}$ and $A \in \Gamma_2$. In this case,

$$\bigcap \Gamma_1 - \bigcup \Gamma_2 \subseteq B - A = C \subseteq \bigcap \Gamma_1 - \bigcup \Gamma_2.$$

So

$$\bigcap \Gamma_1 - \bigcup \Gamma_2 = C.$$

Similarly,

$$\bigcap \Gamma_2 - \bigcup \Gamma_1 = \bigcap F.$$

Since $|C| = |\bigcap F|$, Equality 2.9 holds.

Case b: $B \in \Gamma_1$, $1 < |\Gamma_1|$ and $A \in \Gamma_2$. In this case,

$$\bigcap \Gamma_1 - \bigcup \Gamma_2 = \emptyset = \bigcup \Gamma_2 - \bigcup \Gamma_1.$$

Case c: $\Gamma_1 = \{B\}$ and $A \notin \Gamma_2$. In this case, $\bigcap \Gamma_1 = \bigcup \Gamma_1 = B$ and

$$|B - \bigcup \Gamma_2| = |C| + |A - \bigcup \Gamma_2| = |C| + |\bigcap \Gamma_2 - A| = |\bigcap \Gamma_2 - B|,$$

(the second equality holds, because F is an hke collection). So Equality 2.9 holds in this case, too.

Case d: $B \in \Gamma_1$, $1 < |\Gamma_1|$ and $A \notin \Gamma_2$. In this case, we prove the following claim:

Claim 3.5. Define $\Gamma_3 = \Gamma_1 - \{B\} \cup \{A\}$. The following things hold:

- (1) $\bigcap \Gamma_1 - \bigcup \Gamma_2 = \bigcap \Gamma_3 - \bigcup \Gamma_2$,
- (2) $\bigcap \Gamma_2 - \bigcup \Gamma_1 = \bigcap \Gamma_2 - \bigcup \Gamma_3$ and
- (3) $|\bigcap \Gamma_3 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_3|$.

Proof. (1) Since $\bigcap F \subseteq \bigcup \Gamma_2$, it is enough to show that if $x \in \bigcup F' - \bigcap F$ then $x \in \bigcap \Gamma_1$ if and only if $x \in \bigcap \Gamma_3$. If $x \in C$ then clearly $x \notin \bigcap \Gamma_3$, so x does not belong to the right side. But since $1 < |\Gamma_1|$, x does not belong to $\bigcap \Gamma_1$ too. If $x \notin C$ then clearly, $x \in \bigcap \Gamma_1$ if and only if $x \in \bigcap \Gamma_3$.

(2) Let $x \in \bigcap \Gamma_2$. So $x \notin C$. We show that $x \in \bigcup \Gamma_1$ if and only if $x \in \bigcup \Gamma_3$. If $x \in \bigcap F$ then x belongs to both $\bigcup \Gamma_1$ (because $1 < |\Gamma_1|$) and $\bigcup \Gamma_3$. If $x \notin \bigcap F$ then clearly it belongs to $\bigcup \Gamma_1$ if and only if it belongs to $\bigcup \Gamma_3$.

(3) Since $A \notin \Gamma_2$, the collections Γ_2 and Γ_3 are disjoint subcollections of F . But F is an hke collection. Therefore by Theorem 2.13, Equality 2.9 holds for Γ_2 and Γ_3 . \dashv

Claim 3.5 implies Equality 2.9 (for Γ_1 and Γ_2). Proposition 3.4 is proved. \dashv

Letting an hke collection, F , Proposition 3.6 presents a sufficient and necessary condition for $F \cup \{D\}$ being an hke collection. It is a preparation for Proposition 3.10. In [1] we present an improved version of Proposition 3.6.

We know by Theorem 2.13 that $F \cup \{D\}$ is an hke collection if and only if Equality 2.9 holds for each partition of $F \cup \{D\}$. Proposition 3.6 presents a weaker condition: it is enough to consider partitions of $F \cup \{D\}$ into two subcollections, such that A and D belong to the same subcollection.

Proposition 3.6. *Let F be an hke collection, let $A \in F$ and let D be an arbitrary set. Then $F \cup \{D\}$ is an hke collection if and only if $|D| = \alpha$ and for every partition $\{\Gamma_1, \Gamma_2\}$ of $F - \{A\}$ with $\Gamma_2 \neq \emptyset$, the following holds:*

$$|A \cap D \cap \bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1 - A - D|$$

(if $\Gamma_1 = \emptyset$ then we ignore $\bigcup \Gamma_1$ and $\bigcap \Gamma_1$, or more formally stipulate $\bigcap \Gamma_1 = \bigcup F \cup \{D\}$).

Before proving Proposition 3.6, we present an application for a relevant collection of cardinality 3.

Corollary 3.7. *Let $F' = \{A, B, D\}$ be a relevant collection of cardinality 3. Then F' is an hke collection if and only if $|A \cap D - B| = |B - A - D|$.*

Proof. $F = \{A, B\}$ is an hke collection, because its cardinality is 2. $F - \{A\} = \{B\}$. The unique partition $\{\Gamma_1, \Gamma_2\}$ of $\{B\}$ with $\Gamma_2 \neq \emptyset$ is $\{\emptyset, \{B\}\}$. Now apply Proposition 3.6. \dashv

We now prove Proposition 3.6.

Proof. Let us call the equality above, ‘Equality (*)’.

If $F \cup \{D\}$ is an hke collection then by Proposition 2.13((1) \rightarrow (3)), Equality (*) holds.

Conversely, assume that $|D| = \alpha$ and Equality (*) holds. Applying again Proposition 2.13((3) \rightarrow (1)), it remains to prove that

$$(**) |A \cap \bigcap \Gamma_1 - \bigcup \Gamma_2 - D| = |D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1 - A|,$$

for every partition $\{\Gamma_1, \Gamma_2\}$ of $F - \{A\}$ (including that case $\Gamma_2 = \emptyset$).

Case A: $\Gamma_2 \neq \emptyset$. In this case, we do not use the fact that $|D| = \alpha$. Since F is an hke collection, the following equality holds:

$$|A \cap \bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1 - A|.$$

Hence, by subtracting Equality (*) from the last equality, we obtain Equality (**).

Case B: $\Gamma_2 = \emptyset$. In this case, we have to prove

$$|\bigcap F - D| = |D - \bigcup F|.$$

We use the assumption $|D| = \alpha$ and Case A.

Claim 3.8.

$$|\bigcap \Gamma_1 - \bigcup \Gamma_2 - D| = |D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1|$$

holds, for each partition $\{\Gamma_1, \Gamma_2\}$ of F into two non-empty subcollections.

Proof. By Equality (**) in case A and by Equality (*). \dashv

Consider the following partition of D :

$$\mathbb{D} = \{D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1 : \{\Gamma_1, \Gamma_2\} \text{ is a partition of } F\}.$$

For every set $D' \in \mathbb{D}$, we define a ‘pseudo dual set’, $(D')^d$, that is a subset of A as follows:

$$(D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1)^d = \begin{cases} D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1, & \text{if } A \in \Gamma_2 \\ \bigcap \Gamma_1 - \bigcup \Gamma_2 - D, & \text{if } A \notin \Gamma_2 \end{cases}$$

By Claim 3.8, $|(D')^d| = |D'|$ holds for each $D' \in \mathbb{D}$, except maybe two sets: $D \cap \bigcap F$ and $D - \bigcup F$. But $(D \cap \bigcap F)^d = D \cap \bigcap F$. So the equality $|(D')^d| = |D'|$ holds for $D \cap \bigcap F$ as well. Our goal is to prove this equality for $D - \bigcup F$: to prove that $|(D - \bigcup F)^d| = |D - \bigcup F|$ (namely, $|\bigcap F - D| = |D - \bigcup F|$).

Define

$$\mathbb{A} = \{(D')^d : D' \in \mathbb{D}\}.$$

Since $|D| = \alpha = |A|$, if \mathbb{A} is a partition of A then

$$\sum_{D' \in \mathbb{D}} |D'| = |D| = |A| = \sum_{D' \in \mathbb{D}} |(D')^d|$$

and so the equality $|(D')^d| = |D'|$ holds for $D' = D - \bigcup F$ as needed.

By the following claim, \mathbb{A} is a partition of A :

Claim 3.9. \mathbb{A} satisfies the following things:

- (1) $\bigcup \mathbb{A} \subseteq A$,
- (2) $A \subseteq \bigcup \mathbb{A}$ and
- (3) the pseudo dual sets of each two different sets in \mathbb{D} are disjoint.

Proof. (1) Clear.

(2) Let $x \in A$. Define $\Gamma_2 = \{B \in F : x \in B\}$ and $\Gamma_1 = F - \Gamma_2$. So $x \in \bigcap \Gamma_2 - \bigcup \Gamma_1$ and $A \in \Gamma_2$. If $x \in D$ then $x \in D \cap \bigcap \Gamma_2 - \bigcup \Gamma_1 \in \mathbb{A}$. So $x \in \bigcup \mathbb{A}$.

If $x \notin D$ then $x \in \bigcap \Gamma_2 - \bigcup \Gamma_1 - D \in \mathbb{A}$, because $\bigcap \Gamma_2 - \bigcup \Gamma_1 - D$ is the pseudo dual of $D \cap \bigcap \Gamma_1 - \bigcup \Gamma_2$. So $x \in \bigcup \mathbb{A}$.

(3) Take two different partitions $\{\Gamma_1^a, \Gamma_2^a\}$ and $\{\Gamma_1^b, \Gamma_2^b\}$ of F .

If $A \in \Gamma_2^a - \Gamma_2^b$ then the pseudo dual sets are $D \cap \bigcap \Gamma_2^a - \bigcup \Gamma_1^a$ and $\bigcap \Gamma_1^b - \bigcup \Gamma_2^b - D$. The first is included in D while the second is disjoint to D .

If $A \in \Gamma_2^a \cap \Gamma_2^b$ then the pseudo dual sets are $D \cap \bigcap \Gamma_2^a - \bigcup \Gamma_1^a$ and $D \cap \bigcap \Gamma_2^b - \bigcup \Gamma_1^b$. Since $\Gamma_2^a \neq \Gamma_2^b$, there is a set $B \in \Gamma_2^a - \Gamma_2^b$ or vice versa. So one pseudo dual set is included in B , while the other is disjoint to B .

If $A \notin \Gamma_2^a \cup \Gamma_2^b$ then the pseudo dual sets are $\bigcap \Gamma_1^a - \bigcup \Gamma_2^a - D$ and $\bigcap \Gamma_1^b - \bigcup \Gamma_2^b - D$. So again there is a set $B \in \Gamma_1^a - \Gamma_1^b$ or vice versa. Therefore one pseudo dual set is included in B , while the other is disjoint to B . \dashv

Proposition 3.6 is proved. \dashv

By the following proposition, if F is a maximal hke collection, then for each $A \in F$, f_A is onto the power set of A . Moreover, it specifies several properties of a set D that can be added to F , if f_A is not a surjection.

Proposition 3.10. Let F be an hke collection. Let $E \subseteq A \in F$. Then we can find a set D such that the following holds:

- (1) $A \cap D = E$,
- (2) $F \cup \{D\}$ is an hke collection and
- (3) $|D - \bigcup F| = |\bigcap F - E|$.

In particular, if $\bigcap F = \emptyset$ then $D \subseteq \bigcup F$.

It is recommended to assume $\bigcap F = \emptyset$ at the first reading of the following proof.

Proof. Let us give a rough description of the proof: The needed D should be a set of cardinality α that includes E with $A \cap D = E$. So D will be the union of E with $\alpha - |E|$ elements that are not in A . For each non-empty subset Γ of $F - \{A\}$ the ‘dual set’ has the same number of elements. But we subtract e_Γ elements from the dual set, so that at the end, we will subtract $|E|$ elements. But if $\Gamma = \emptyset$ then there is no dual set and we have to choose elements outside of $\bigcup F$.

For every subset Γ of F such that $A \in \Gamma$, we define e_Γ as follows: Define $\Gamma_1 = \Gamma$ and $\Gamma_2 = F - \Gamma_1$. Define

$$e_\Gamma =: |E \cap \bigcap \Gamma_1 - \bigcup \Gamma_2|.$$

Since $E \subseteq A$, the following equality holds:

$$(1) \quad \sum_{A \in \Gamma \subseteq F} e_\Gamma = |E|.$$

By Proposition 2.13, if Γ_2 is not empty (or equivalently, $\Gamma \neq F$) then

$$|\bigcap \Gamma_2 - \bigcup \Gamma_1| = |\bigcap \Gamma_1 - \bigcup \Gamma_2| \geq e_\Gamma.$$

We choose a set, E_Γ , of e_Γ elements in $\bigcap \Gamma_2 - \bigcup \Gamma_1$ for each such Γ (E_Γ is the set of elements in $\bigcap \Gamma_2 - \bigcup \Gamma_1$ that are not going to be in the needed set, D).

Let C be a set of cardinality $|\bigcap F| - e_F = |\bigcap F - E|$ that is disjoint to $\bigcup F$. Define

$$D =: E \cup \left[\bigcup F - A - \bigcup_{A \in \Gamma \subsetneq F} E_\Gamma \right] \cup C.$$

Clearly $D \cap A = E$.

It remains to show that $F \cup \{D\}$ is an hke collection. In order to apply Proposition 3.6, we firstly have to prove that $|D| = \alpha$.

$$|D| = |E| + \left| \bigcup F - A \right| - \sum_{A \in \Gamma \subsetneq F} e_\Gamma + |C|.$$

By Equality (1) and the equality $|C| = |\bigcap F| - e_F$, we get

$$|D| = |E| + \left| \bigcup F - A \right| - \sum_{A \in \Gamma \subsetneq F} e_\Gamma + |\bigcap F| = \left| \bigcup F \right| + |\bigcap F| - |A| = 2\alpha - \alpha = \alpha.$$

The equality $|D| = \alpha$ is proved. It remains to show that the second condition of Proposition 3.6 is satisfied.

Let $\{\Gamma_1, \Gamma_2\}$ be a partition of F such that $A \in \Gamma_1$ and $\Gamma_2 \neq \emptyset$.

We have to show that

$$|D \cap \bigcap \Gamma_1 - \bigcup \Gamma_2| = |\bigcap \Gamma_2 - \bigcup \Gamma_1 - D|.$$

Define $\Gamma = \Gamma_1$. Since $D \cap A = E$ and $A \in \Gamma_1$,

$$|D \cap \bigcap \Gamma_1 - \bigcup \Gamma_2| = |E \cap \bigcap \Gamma_1 - \bigcup \Gamma_2| = e_\Gamma.$$

It remains to show that

$$|\bigcap \Gamma_2 - \bigcup \Gamma_1 - D| = e_\Gamma.$$

By the definition of D and the fact that the sets E and C are disjoint to the set $\bigcap \Gamma_2 - \bigcup \Gamma_1$, we have

$$\bigcap \Gamma_2 - \bigcup \Gamma_1 - D = \bigcap \Gamma_2 - \bigcup \Gamma_1 - \left(\bigcup F - A - \bigcup_{A \in \Gamma' \subsetneq F} E_{\Gamma'} \right).$$

Since $\bigcap \Gamma_2 - \bigcup \Gamma_1 \subseteq \bigcup F - A$, the right side of the last equality equals

$$\bigcap \Gamma_2 - \bigcup \Gamma_1 - \left(\bigcap \Gamma_2 - \bigcup \Gamma_1 - \bigcup_{A \in \Gamma' \subsetneq F} E_{\Gamma'} \right),$$

namely,

$$\left(\bigcap \Gamma_2 - \bigcup \Gamma_1 \right) \cap \bigcup_{A \in \Gamma' \subsetneq F} E_{\Gamma'} = E_{\Gamma}$$

(the last equality holds, because for each $\Gamma' \neq \Gamma$, the sets $E_{\Gamma'}$ and $\bigcap \Gamma_2 - \bigcup \Gamma_1$ are disjoint).

Hence,

$$|\bigcap \Gamma_2 - \bigcup \Gamma_1 - D| = |E_{\Gamma}| = e_{\Gamma}.$$

–

Remark 3.11. We now can present a new proof of Proposition 3.4. Assume that $\bigcap F \neq \emptyset$. Define $E = \emptyset$. Let $A \in F$. $E \subseteq A$ is not in the image of f_A , because $A \cap D = \bigcap F \neq \emptyset$ for each $D \in F$. So f_A is not a surjection and by Proposition 3.10, F is not a maximal hke collection.

Theorem 3.12. *Let F be an hke collection. Then F is a maximal hke collection if and only if $|F| = 2^\alpha$.*

Proof. If $|F| = 2^\alpha$ then by Proposition 3.3, F is a maximal hke collection.

Conversely, assume that F is a maximal hke collection. Fix $A \in F$. By Propositions 3.2 and 3.10, $f_A : F \rightarrow P(A)$ is a bijection. So $|F| = 2^\alpha$. –

4. THE UNIQUENESS OF A MAXIMAL HKE COLLECTION

In this section (Theorem 4.2), we present two similar characterizations of a maximal hke collection and use it to prove the uniqueness of a maximal hke collection for a fixed α (Theorem 4.9). Moreover, this characterization yields a connection between maximal hke *collections* and minimal KE *graphs* (Theorem 5.11).

The following definition is needed in order to state Theorem 4.2.

Definition 4.1. Let F be an hke collection. We define a relation \approx on $\bigcup F$ as follows: $x \approx y$ if and only if $x = y$ or $A - D = \{x\}$ and $D - A = \{y\}$ for some $A, D \in F$. If F is not clear from the context, we will write \approx_F . If F is a maximal hke collection, $x \neq y$ and $x \approx y$ then y is said to be *the dual* of x (this name is justified in the proof of Theorem 4.2). If F is not clear from the context, we will write ‘the F -dual’.

Clauses (2) and (3) of Theorem 4.2 are similar. In Clause (2), we require the existence of an equivalence relation satisfying several specific properties. In Clause (3), we require that \approx will satisfy these properties.

Theorem 4.2. *Let F be a relevant collection with $\alpha(F) = \alpha$. The following things are equivalent:*

- (1) F is a maximal hke collection.
- (2) $|\bigcup F| = 2\alpha$ and there is an equivalence relation on $\bigcup F$ with α equivalence classes, each has two elements such that F equals the collection of subsets B of $\bigcup F$ such that $|B| = \alpha$ and B intersects each equivalence class.

- (3) $|\bigcup F| = 2\alpha$, \approx has α equivalence classes, each equivalence class has two elements and F equals the collection of subsets B of $\bigcup F$ such that $|B| = \alpha$ and B intersects each \approx -equivalence class.

Proof. Clause (3) implies Clause (2) trivially. Assume that Clause (2) holds. By renaming, without loss of generality, $\bigcup F = [2\alpha]$ and the equivalence classes are the pairs of the form $\{m, m + \alpha\}$ for $1 \leq m \leq \alpha$. Then F is the typical collection for α . So by Proposition 2.8, F is an hke collection and $|F| = 2^\alpha$. So by Proposition 3.3, Clause (1) holds.

It remains to prove that Clause (1) implies Clause (3). Assume that F is a maximal hke collection. In Claims 4.3-4.7, we prove that \approx is an equivalence relation and study its properties.

Claim 4.3. *For every $x \in \bigcup F$, there are two sets A and D in F such that $A - D = \{x\}$.*

Proof. Take $A \in F$ such that $x \in A$. By Proposition 3.10, f_A is a surjection. So there is a set $D \in F$ such that $A \cap D = A - \{x\}$. Therefore $A - D = \{x\}$. \dashv

By Claim 4.6 (below), the relation \approx is an equivalence relation. By the following claim, each \approx -equivalence class has two elements at least.

Claim 4.4. *For every element $x \in \bigcup F$, there is an element $y \in \bigcup F$ such that $x \approx y$ and $x \neq y$.*

Proof. By Claim 4.3, there are two sets, A and D in F , such that $A - D = \{x\}$. By Theorem 2.13, $|D - A| = |A - D| = 1$. Let y be the unique element in $D - A$. Then $x \approx y$. \dashv

Claim 4.5. *Let x, y be two different elements in $\bigcup F$. Then $x \approx y$ if and only if $x \in B \leftrightarrow y \notin B$ holds for every $B \in F$.*

Proof. First assume that $x \in B \leftrightarrow y \notin B$ holds for every $B \in F$. By Claim 4.3, there are two sets A and D in F , such that $A - D = \{x\}$. So $|D - A| = |A - D| = 1$. But $y \in A - D$. Therefore $D - A = \{y\}$ and $x \approx y$.

Conversely, assume that $x \approx y$. So there are A and D in F such that $A - D = \{x\}$ and $D - A = \{y\}$. Let $B \in F$. We show that $x \in B$ if and only if $y \notin B$ using the following three equalities:

- (1) $A \cap B - D = (A - D) \cap B = \{x\} \cap B$,
- (2) $|A \cap B - D| = |D - A - B|$ (By Theorem 2.13) and
- (3) $D - A - B = \{y\} - B$.

$x \in B$ if and only if $\{x\} \cap B \neq \emptyset$ if and only if $|A \cap B - D| \neq 0$ if and only if $|D - A - B| \neq 0$ if and only if $\{y\} - B \neq \emptyset$, namely $y \notin B$. \dashv

By the following claim, the relation \approx is transitive, so it is an equivalence relation. Moreover, each equivalence class has two elements at most.

Claim 4.6. *If $x \neq y$, $x \approx y$, $y \neq z$ and $y \approx z$ then $x = z$.*

[Here is another formulation of Claim 4.6: if x is the dual of y and z is the dual of y then $x = z$.]

Proof. By Claim 4.5, for every $B \in F$, we have

$$x \in B \Leftrightarrow y \notin B \Leftrightarrow z \in B.$$

By Claim 4.3, there are two sets, $A, D \in F$ such that $A - D = \{x\}$. Since $x \in A - D$, we have $z \in A - D$, as well. So $z \in \{x\}$, namely, $z = x$. \dashv

Claim 4.7. *The relation \approx is an equivalence relation. It has α equivalence classes, each has two elements exactly.*

Proof. By its definition, the relation \approx is reflexive and symmetric. By Claim 4.6, \approx is transitive. So it is an equivalence relation. By Claim 4.4, each equivalence class of \approx has two elements at least. By Claim 4.6 each equivalence class of \approx has two elements at most. So each equivalence class of \approx has exactly two elements.

We now show that there are exactly α \approx -equivalence classes. Fix $A \in F$. Let f be the function of the set of \approx -equivalence classes to A , such that $f(\{x, y\})$ is the unique element in $\{x, y\} \cap A$. The number of \approx -equivalence classes is α , because f is a bijection: Let $\{x, y\}$ be an \approx -equivalence class. By Claim 4.5, $x \notin A$ if and only if $y \in A$. So f is well-defined. Since each $x \in A$ is in some \approx -equivalence class, f is surjective. Since an element in A cannot be in two different \approx -equivalence classes at the same time, f is injective. \dashv

Let F' be the collection of subsets B of $\bigcup F$ such that $|B| = \alpha$ and B intersects each equivalence class of \approx . By Claim 4.7, F' is the collection of subsets B of $\bigcup F$ such that $|B \cap \{x, y\}| = 1$ for each equivalence class $\{x, y\}$ of \approx . So $|F'| = 2^\alpha$.

It remains to prove that $F' = F$. Since F is a maximal hke collection, by Theorem 3.12, $|F| = 2^\alpha$ too. So it is enough to prove that $F \subseteq F'$. Let $B \in F$. So $|B| = \alpha$. Let $\{x, y\}$ be an \approx -equivalence relation. Since $x \approx y$, By Claim 4.5, $x \in B$ if and only if $y \notin B$. So $\{x, y\} \cap B \neq \emptyset$. Therefore $B \in F'$. Hence, $F' = F$. Clause (3) is proved. So Theorem 4.2 is proved. \dashv

Definition 4.8. Let F_1 and F_2 be two collections of sets. We say that F_1 and F_2 are *isomorphic* when: there is a bijection $g : \bigcup F_1 \rightarrow \bigcup F_2$ such that $g[S] \in F_2$ for every $S \in F_1$ and $g^{-1}[S] \in F_1$ for every $S \in F_2$.

By the following theorem, up to isomorphism, there is a unique maximal hke collection with α fixed:

Theorem 4.9. *If F_1 and F_2 are two maximal hke collections with $\alpha(F_1) = \alpha(F_2)$ then $F_1 \cong F_2$.*

Proof. Define $\alpha = \alpha(F_1) = \alpha(F_2)$. Fix $A_1 \in F_1$ and $A_2 \in F_2$. So $|A_1| = |A_2| = \alpha$. Let $f : A_1 \rightarrow A_2$ be a bijection. Extend f to a function $g : \bigcup F_1 \rightarrow \bigcup F_2$ such that if y is the dual of x in F_1 then $g(y)$ is the dual of $g(x)$ in F_2 (it holds vacuously for $x, y \in A_1$, because in this case, y is not the dual of x). So for every two elements $x, y \in \bigcup F_1$, y is the dual of x in F_1 if and only if $g(y)$ is the dual of $g(x)$ in F_2 . By Claim 4.7, g is bijective.

By the symmetry between F_1 and F_2 , in order to prove that g is an isomorphism, it is enough to show that $g[S]$ belongs to F_2 for each $S \in F_1$. By Theorem 4.2((1) \rightarrow (3)), for every subset B of $\bigcup F_2$ the following holds: $B \in F_2$ if and only if $\alpha \leq |B|$ and each two different elements in B are not \approx_{F_2} -equivalent.

Since g is injective, $\alpha = |S| \leq g[S]$. Let $g(x), g(y)$ be two different elements in $g[S]$. If $g(x) \approx_{F_2} g(y)$ then $g(y)$ is the dual of $g(x)$ and so y is the dual of x . But y cannot be the dual of x , because they are both in S . \dashv

Corollary 4.10. *Let F be a relevant collection with $\alpha(F) = \alpha$. Then F is a maximal hke collection if and only if F is isomorphic to the typical collection for α .*

Proof. By Theorem 4.9, we have only to prove that if F is the typical collection for α then F is a maximal hke collection. Assume that F is the typical collection for α . By Proposition 2.8, F is an hke collection and $|F| = 2^\alpha$. So by Theorem 3.12, F is a maximal hke collection. \dashv

The following corollary will be used in the proof of Theorem 5.11.

Corollary 4.11. *Let F be a maximal hke collection. Let x and y be two different elements in $\bigcup F$. Then the following conditions are equivalent:*

- (1) $x \approx y$,
- (2) $\{x, y\} \cap S \neq \emptyset$ for each $S \in F$ and
- (3) $\{x, y\} \not\subseteq S$ for each $S \in F$.

Proof. By Corollary 4.10, without loss of generality, F is the typical collection for α (in particular, x and y are numbers in $[2\alpha]$). So each condition ((1), (2) and (3)) is equivalent to the condition $|x - y| = \alpha$. \dashv

5. FROM A COLLECTION TO A GRAPH

Definition 5.1. Let F be a relevant collection. The *graph of F* , $G(F)$, is the graph $(V(G), E(G))$, where $V(G) =: \bigcup F$ and $E(G) =: \{vu : v, u \in G \text{ and there is no } A \in F \text{ such that } \{u, v\} \subseteq A\}$.

$G(F)$ has the maximal set of edges such that each set in F is independent.

Proposition 5.2. *Let F be a relevant collection. Then $\alpha(F) \leq \alpha(G(F))$.*

Proof. Every set in F is independent in $G(F)$. Therefore $\alpha(F) \leq \alpha(G(F))$. \dashv

It is easy to find a relevant collection, F , such that $\alpha(F) < \alpha(G(F))$.

Example 5.3.

$$F := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

Here, $\alpha(F) = 2$, while $\alpha(G(F)) = 3$.

Example 5.4.

$$F := \{\{0, 1, 2, 3, 4, 8\}, \{0, 3, 4, 5, 6, 7\}, \{0, 1, 2, 5, 6, 9\}\}$$

Here, $\alpha(F) = 6$, while $\alpha(G(F)) = 7$.

Recall:

Definition 5.5. A *well-covered graph* is a graph in which every independent set can be extended to a maximum independent set.

Proposition 5.6. *For every well-covered graph G with $V(G) = \text{corona}(G)$, the graph of $\Omega(G)$ is G .*

Proof. Let G be a well-covered graph with $V(G) = \text{corona}(G)$ and let G' be the graph of $\Omega(G)$. We should prove that $G = G'$. They have the same set of vertices:

$$V(G') = \bigcup \Omega(G) = \text{corona}(G) = V(G).$$

They have the same set of edges as well: First assume that $(uv) \in E(G)$. $\{u, v\} \not\subseteq X$, for every $X \in \Omega(G)$. So $(uv) \in E(G')$. Conversely, assume that $(uv) \notin E(G)$. So $\{u, v\}$ is an independent set in G . Since G is well-covered, we can find a set $X \in \Omega(G)$ such that $\{u, v\} \subseteq X$. Therefore $(uv) \notin E(G')$. \dashv

The next example exemplifies the following facts:

- (1) there is an hke collection F such that $\Omega(G) \neq F$ for every graph G and
- (2) we can find two different hke collections, F_1 and F_2 such that $G(F_1) = G(F_2)$.

Example 5.7.

$$F := \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 5\}, \{2, 4, 5\}, \{2, 4, 6\}\}.$$

F is an hke collection and $G(F)$ is the bipartite graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{12, 34, 36, 56\}$. But

$$\Omega(G) = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 5\}, \{2, 4, 5\}, \{2, 4, 6\}, \{1, 4, 5\}\} = F \cup \{1, 4, 5\} \neq F.$$

Actually, there is no graph G with $\Omega(G) = F$. For let G be a graph with $F \subseteq \Omega(G)$. So $E(G) \cap \{45, 14, 15\} = \emptyset$. But in this case, $\{1, 4, 5\} \in \Omega(G) - F$.

The graph of $\Omega(G)$ is G too. So $F_1 = F$ and $F_2 = \Omega(G)$ exemplify Fact (2).

Example 5.8.

$$F = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}\}.$$

$\alpha(F) = 2$, $\bigcup F = \{1, 2, 3, 4\}$ and $\bigcap F = \emptyset$. But F is not an hke collection, because its subcollection, $\Gamma = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, is not a KE collection. Let G be the graph of F . $\alpha(G) = 3$ (because $\{1, 2, 3\} \in \Omega(G)$) and G is not a KE graph.

Problem 5.9. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two KE graphs. We say that G_1 and G_2 are *equivalent* when $\Omega(G_1) \cong \Omega(G_2)$. Find connections between equivalent graphs.

Definition 5.10. The *typical KE graph* for α is the following bipartite graph: $V(G) = [2\alpha]$ and $E(G) = \{i, i + \alpha : i \in [\alpha]\}$.

Theorem 5.11. Let G be a graph such that $V(G) = \text{corona}(G)$. Then G is isomorphic to the typical KE graph for $\alpha(G)$ if and only if $\Omega(G)$ is a maximal hke collection.

Proof. Assume that G is isomorphic to the typical KE graph for α . Without loss of generality $V(G) = [2\alpha]$ and $E(G) = \{i, i + \alpha : i \in [\alpha]\}$, for some α . So a maximal independent set in G has α elements. Clearly, $\Omega(G)$ is the typical collection for α . Hence, by Corollary 4.10, $\Omega(G)$ is a maximal hke collection.

Conversely, assume that $\Omega(G)$ is a maximal hke collection. By Theorem 4.10, without loss of generality, $\Omega(G)$ is the typical collection for α . So $\text{corona}(G) = \bigcup \Omega(G) = [2\alpha]$. Therefore $V(G) = \text{corona}(G) = [2\alpha]$.

By Definition 5.10, it remains to prove that $xy \in E(G)$ if and only if $|x - y| = \alpha$, or equivalently, x is the $\Omega(G)$ -dual of y .

Claim 5.12. Let x and y be two different vertices. If $xy \in E(G)$ then x is the $\Omega(G)$ -dual of y .

Proof. For every $S \in \Omega(G)$, S is independent and so the set $\{x, y\}$ is not included in S . Hence, by Corollary 4.11, x is the $\Omega(G)$ -dual of y . \dashv

Claim 5.13. *Let x and y be two different vertices. Then $xy \in E(G)$ if and only if x is the $\Omega(G)$ -dual of y .*

Proof. The first direction holds by Claim 5.12. In order to prove the second direction, we assume that $xy \notin E(G)$. For the sake of a contradiction, assume that x is the $\Omega(G)$ -dual of y . So by the uniqueness of the $\Omega(G)$ -dual of y , for every $z \in V(G) - \{x, y\}$, z is not the $\Omega(G)$ -dual of y . By Claim 5.12, $zy \notin E(G)$. But by assumption, $xy \notin E(G)$. So for every $z \in V(G) - \{y\}$, $zy \notin E(G)$.

Take $S \in \Omega(G)$ such that $x \in S$. Since $x \approx y$, by Corollary 4.11, $y \notin S$. Hence, $S \cup \{y\}$ is an independent set of cardinality $\alpha + 1$, a contradiction. \dashv

Theorem 5.11 is proved. \dashv

6. THE SUBCOLLECTIONS OF $\Omega(G)$ WHERE G IS KE

We now can solve a problem of Jarden, Levit and Mandrescu.

The definition of a KE collection in [2], is different from the definition in the current paper.

Definition 6.1. A relevant collection, F , is said to be a *KE collection in the old sense* if $F \subseteq \Omega(G)$, for some graph G and $|\bigcup F| + |\bigcap F| = 2\alpha(G)$.

We restate [2, Problem 3.1]:

Problem 6.2. Characterize the KE collections in the old sense.

Theorem 6.3. *Let F be a collection of sets. The following conditions are equivalent:*

- (1) F is an hke collection,
- (2) F is isomorphic to a subcollection of the typical collection for $\alpha(F)$,
- (3) F is included in $\Omega(G)$ for some KE graph G and
- (4) F is a KE collection in the old sense.

Proof. We first prove that Clause (1) implies clause (2). Assume that F is an hke collection. Let F' be a maximal hke collection including F . Clearly, $\alpha(F') = \alpha(F)$. Let G be the typical graph for $\alpha(F)$. By Theorem 5.11, $\Omega(G)$ is a maximal hke. So by Theorem 4.9, F' is isomorphic to $\Omega(G)$. Clause (2) is proved.

Easily, Clause (2) implies Clause (3).

If Clause (3) holds then

$$|\bigcup F| + |\bigcap F| \leq |\bigcup \Omega(G)| + |\bigcap \Omega(G)| = 2\alpha.$$

So Clause (4) holds.

Assume that Clause (4) holds, namely, F is a KE collection in the old sense. So $F \subseteq \Omega(G)$ for some graph G and

$$|\bigcup F| + |\bigcap F| = 2\alpha(G).$$

Let Γ be a subcollection of F . By [4, Corollary 2.7 and Corollary 2.9],

$$|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G).$$

\dashv

7. ADDING A LIMITATION ON $|\bigcup F|$

In this section, we prove that for every two positive integers α and n , the maximal cardinality of an hke collection, F , with $\alpha(F) = \alpha$ and $|\bigcup F| = n$ is $2^{n-\alpha}$. We conclude that for every positive integer n , the maximal cardinality of an hke collection, F , with $|\bigcup F| = n$ is 2^α if $n = 2\alpha$ or $n = 2\alpha + 1$.

Since every hke collection F satisfies $\alpha(F) \leq |\bigcup F| \leq 2\alpha(F)$, the assumption $\alpha \leq n \leq 2\alpha$ in the following definition is needed.

Definition 7.1. Let $a(\alpha, n)$ be the maximal number of sets in an hke collection F with $\alpha(F) = \alpha$ and $|\bigcup F| = n$, where α and n are positive integers satisfying $\alpha \leq n \leq 2\alpha$.

Proposition 7.2. For every two positive integers α and n satisfying $\alpha \leq n \leq 2\alpha$, we have $a(\alpha, n) \leq 2^\alpha$.

Proof. By Theorem 3.12. ◄

Proposition 7.3. For every positive integer α , we have $a(\alpha, 2\alpha) = 2^\alpha$.

Proof. By Proposition 7.2, $a(\alpha, 2\alpha) \leq 2^\alpha$. The typical example for α exemplifies the converse. ◄

In order to prove that $a(\alpha, n) = 2^{n-\alpha}$, it remains to prove that $a(\alpha, n) = a(n - \alpha, 2n - 2\alpha)$. Propositions 7.4 and 7.5 are steps towards this goal.

Proposition 7.4. Let α, n and d be positive integers with $\alpha \leq n \leq 2\alpha$.

- (1) $a(\alpha, n) \leq a(\alpha - d, n - d)$ whenever $a(\alpha - d, n - d)$ is defined (namely, $d \leq 2\alpha - n$) and
- (2) $a(\alpha, n) \leq a(\alpha + d, n + d)$.

Proof. Let F be an hke collection with $\alpha(F) = \alpha$ and $|\bigcup F| = n$.

- (1) $|\bigcap F| = 2\alpha - n$. Let C be a subset of $\bigcap F$ of cardinality d . Define

$$F' = \{A - C : A \in F\}.$$

Since $|\bigcup F'| = n - d$, $\alpha(F') = \alpha - d$ and $|F'| = |F|$, it remains to prove that F' is an hke collection.

Let Γ' be a non-empty subcollection of F' . So

$$\Gamma' = \{A - C : A \in \Gamma\},$$

for some non-empty subcollection Γ of F . We have to show that Γ' is a KE collection, or equivalently,

$$|\bigcup \Gamma'| + |\bigcap \Gamma'| = 2(\alpha - d).$$

But

$$\bigcup \Gamma' = \bigcup \Gamma - C$$

and

$$\bigcap \Gamma' = \bigcap \Gamma - C.$$

So

$$|\bigcup \Gamma'| + |\bigcap \Gamma'| = |\bigcup \Gamma| - d + |\bigcap \Gamma| - d = 2\alpha - 2d.$$

(2) Define $F' = \{A \cup C : A \in F\}$, where C is a fixed set of cardinality d with $C \cap \bigcup F = \emptyset$. Since $|\bigcup F'| = n + d$, $\alpha(F') = \alpha + d$ and $|F'| = |F|$, it remains to prove that F' is an hke collection.

Let Γ' be a non-empty subcollection of F' . So

$$\Gamma' = \{A \cup C : A \in \Gamma\},$$

for some non-empty subcollection Γ of F . We have to show that Γ' is a KE collection, or equivalently,

$$|\bigcup \Gamma'| + |\bigcap \Gamma'| = 2(\alpha + d).$$

But

$$\bigcup \Gamma' = \bigcup_{\Gamma \cup C}$$

and

$$\bigcap \Gamma' = \bigcap_{\Gamma \cup C}.$$

So

$$|\bigcup \Gamma'| + |\bigcap \Gamma'| = |\bigcup \Gamma| + d + |\bigcap \Gamma| + d = 2\alpha + 2d.$$

–

Proposition 7.5. *Let α and n be positive integers satisfying $\alpha \leq n \leq 2\alpha$. If $-\infty < d \leq 2\alpha - n$, then*

$$a(\alpha, n) = a(\alpha - d, n - d).$$

Proof. By Proposition 7.4, it is enough to show that $a(\alpha - d, n - d)$ is defined. The assumption $d \leq 2\alpha - n$ is equivalent to $\alpha' \leq n' \leq 2\alpha'$ where $\alpha' = \alpha - d$ and $n' = n - d$ [$d \leq 2\alpha - n$ yields $n' \leq 2\alpha'$, for $2\alpha' = 2\alpha - 2d = 2\alpha - d - d \geq 2\alpha - d + (n - 2\alpha) = n - d = n'$]. So $a(\alpha - d, n - d)$ is defined. –

Theorem 7.6.

$$a(\alpha, n) = 2^{n-\alpha}$$

holds for each two positive integers α and n satisfying $\alpha \leq n \leq 2\alpha$.

Proof. By Proposition 7.5 (where $2\alpha - n$ stands for d), we have

$$a(\alpha, n) = a(n - \alpha, 2n - 2\alpha).$$

But by Proposition 7.3,

$$a(n - \alpha, 2n - 2\alpha) = 2^{n-\alpha}.$$

–

Let $\lceil n \rceil$ denote the integral value of n .

Corollary 7.7. *Let n be a positive integer. The maximal cardinality of an hke collection, F , with $|\bigcup F| = n$ is $2^{\lceil \frac{n}{2} \rceil}$.*

Proof. Let $c(n)$ be the maximal cardinality of an hke collection, F , with $|\bigcup F| = n$.

$$c(n) = \max\{a(\alpha, n) : \alpha \leq n \leq 2\alpha\}.$$

By Theorem 7.6, $a(\alpha, n) = 2^{n-\alpha}$. So $a(\alpha, n)$ is maximal, when α is the minimal integer such that $n \leq 2\alpha$, or equivalently, $\alpha = \lceil \frac{n+1}{2} \rceil$. Therefore $c(n) = a(\lceil \frac{n+1}{2} \rceil, n)$. If $n = 2k$ then $c(n) = a(\lceil \frac{n+1}{2} \rceil, n) = a(k, 2k) = 2^k$ and if $n = 2k + 1$ then $c(n) = a(\lceil \frac{n+1}{2} \rceil, n) = a(k + 1, 2k + 1) = a(k, 2k) = 2^k$. In any case, $c(n) = 2^{\lceil \frac{n}{2} \rceil}$. –

Connections between the current paper, [3], [5] and [6] should be studied.

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E-mail address, Adi Jarden: `jardena@ariel.ac.il`

DEPARTMENT OF MATHEMATICS., ARIEL UNIVERSITY, ARIEL, ISRAEL